STATISTICS II

Bachelor's degrees in Economics, Finance and Management

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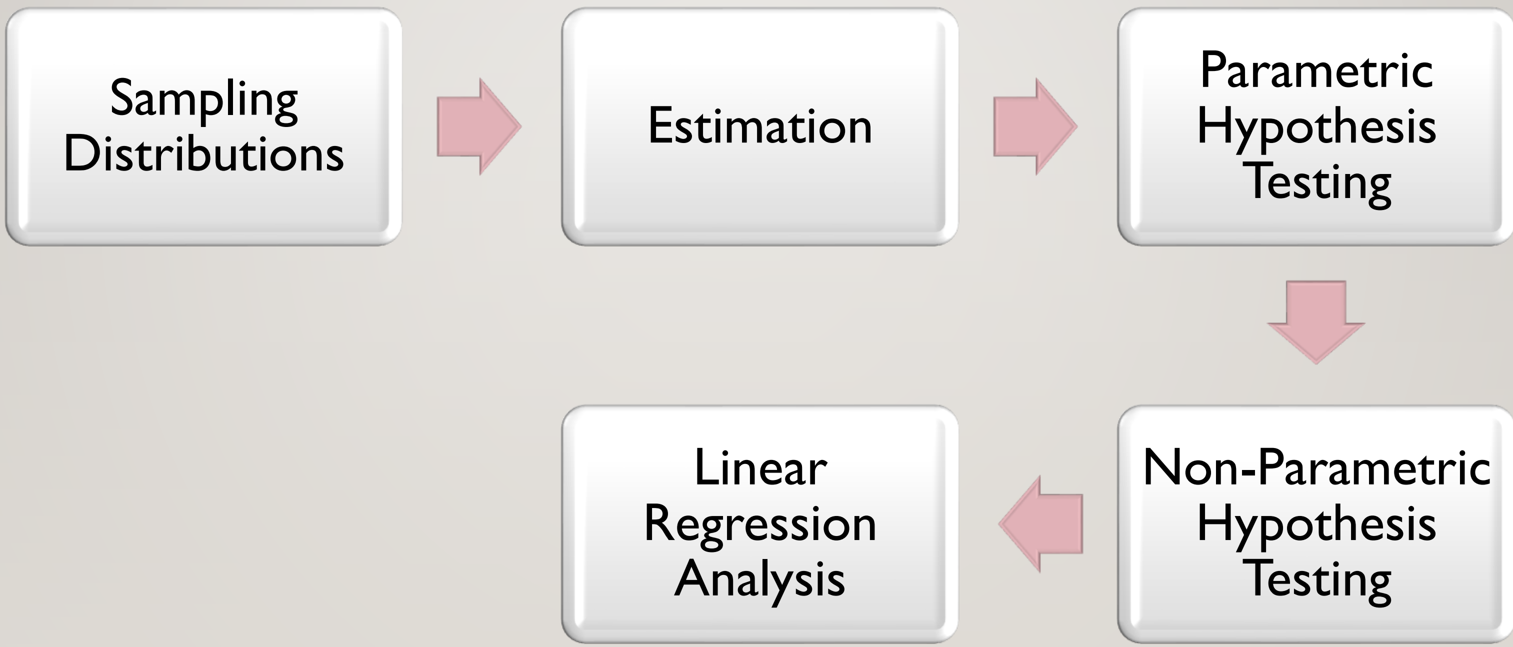
PROGRAM


Sampling Distributions
Estimation
Parametric Hypothesis Testing
Non-Parametric Hypothesis Testing
Linear Regression Analysis

LECTURE 11: ESTIMATION METHODS – MAXIMUM LIKELIHOOD METHOD

MAXIMUM LIKELIHOOD METHOD (MLM)

This method can only be applied if the population distribution is known.

Definition: Let $X_1, X_2, \ldots, X_n$ be a random sample from a given population with probability (density) function

$$f(x; \theta_1, \theta_2, \ldots, \theta_k) = f(x; \theta).$$

Then the joint probability (density) function of the sample variables is given by:

$$f(x_1, x_2, \ldots, x_n; \theta) = f(x_1; \theta) f(x_2; \theta) \ldots f(x_n; \theta) = \prod_{i=1}^{n} f(x_i; \theta).$$

For a given sample, the function of θ is called the **likelihood function**:

$$\mathcal{L}(\theta) = \mathcal{L}(\theta; x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f(x_i; \theta).$$

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MAXIMUM LIKELIHOOD METHOD (MLM)

The **maximum likelihood method** consists of finding the estimator $\hat{\theta}$ that maximizes the value of the likelihood function for a given sample; that is, the value of θ that makes the observed sample most probable, i.e., most **likely**.

Frequently, the maximum likelihood estimator can be found by derivation, following these steps:

1. Determine the likelihood function $\mathcal{L}(\theta)$.
2. If necessary, apply the logarithmic transformation to the likelihood function, $\ln(\mathcal{L}(\theta))$. This transformation often simplifies the maximization problem.
3. Find the points where the **first derivative** of the function $\mathcal{L}(\theta)$, or $\ln(\mathcal{L}(\theta))$, with respect to each θ_i vanishes (first-order condition):

$$\frac{\partial \mathcal{L}(\theta)}{\partial \theta_i} = 0 \quad \text{or} \quad \frac{\partial \ln(\mathcal{L}(\theta))}{\partial \theta_i} = 0.$$

4. Verify that the **second derivative** of the function $\mathcal{L}(\theta)$, or $\ln(\mathcal{L}(\theta))$, with respect to θ_i is negative (second-order condition):

$$\frac{\partial^2 \mathcal{L}(\theta)}{\partial \theta_i^2} < 0 \quad \text{or} \quad \frac{\partial^2 \ln(\mathcal{L}(\theta))}{\partial \theta_i^2} < 0.$$

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MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). . Determine a **maximum likelihood estimate** of λ.

Step 0: Problem

We have $X \sim \text{Poisson}(\lambda)$ and a sample:

$$0, 0, 2, 5, 3, 1$$

We want the **MLE of λ**.

Step 1: Likelihood function

The **Poisson pmf** is:

$$P(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

The **likelihood function** for the sample is:

$$L(\lambda) = \prod_{i=1}^{6} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{\lambda^{\sum x_i} e^{-6\lambda}}{\prod x_i!}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). Determine a **maximum likelihood estimate** of λ.

Step 2: Log-likelihood

Take the natural logarithm:

$$\ell(\lambda) = \ln L(\lambda) = \sum_{i=1}^{6} x_i \ln \lambda - 6\lambda - \sum_{i=1}^{6} \ln(x_i!)$$

$$\ell(\lambda) = \left(\sum x_i\right) \ln \lambda - 6\lambda - \sum \ln(x_i!)$$

Compute $\sum x_i$:

$$0 + 0 + 2 + 5 + 3 + 1 = 11$$

So:

$$\ell(\lambda) = 11 \ln \lambda - 6\lambda - \sum \ln(x_i!)$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 1

Consider the random variable $X \sim Poisson(\lambda)$ and the sample (0, 0, 2, 5, 3, 1). Determine a **maximum likelihood estimate** of λ.

1. $y = u^n \Rightarrow y' = nu^{n-1}u'$;
2. $y = c \Rightarrow y' = 0$, onde k é uma constante real;
3. $y = uv \Rightarrow y' = u'v + v'u$
4. $y = \frac{u}{v} \Rightarrow y' = \frac{u'v - v'u}{v^2}$
5. $y = a^u \Rightarrow y' = a^u(\ln a)u', (a > 0, a \neq 1)$
6. $y = e^u \Rightarrow y' = e^u u'$
7. $y = \log_a u \Rightarrow y' = \frac{u'}{u} \log_a e$
8. $y = \ln u \Rightarrow y' = \frac{1}{u} u'$
9. $y = u^v \Rightarrow y' = vu^{v-1}u' + u^v(\ln u)v'$
10. $y = \sin u \Rightarrow y' = u' \cos u$
11. $y = \cos u \Rightarrow y' = -u' \sin u$
12. $y = \tan u \Rightarrow y' = u' \sec^2 u$, desde que $x \neq (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$;
13. $y = \cot u \Rightarrow y' = -u' \csc^2 u$, desde que $x \neq n\pi, n \in \mathbb{Z}$;

Step 3: First derivative

Differentiate with respect to λ:

$$\frac{d\ell}{d\lambda} = \frac{11}{\lambda} - 6$$

Set derivative to zero:

$$\frac{11}{\hat{\lambda}} - 6 = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{11}{6} \approx 1.833$$

Step 4: Second derivative

$$\frac{d^2\ell}{d\lambda^2} = -\frac{11}{\lambda^2} < 0$$

- Negative confirms **maximum**.

Answer

$$\boxed{\hat{\lambda} = 11/6 \approx 1.833}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

Step 0: Problem

We have $X \sim \text{Exponential}(\lambda)$ and a sample:

$$1.2, 0.5, 3$$

We want the **maximum likelihood estimate (MLE) of** λ.

Step 1: Likelihood function

The pdf of the exponential distribution is:

$$f(x;\lambda) = \lambda e^{-\lambda x}, \quad x \geq 0$$

For the sample, the **likelihood function** is:

$$L(\lambda) = \prod_{i=1}^{3} \lambda e^{-\lambda x_i} = \lambda^3 e^{-\lambda \sum x_i}$$

$$\sum x_i = 1.2 + 0.5 + 3 = 4.7$$

So:

$$L(\lambda) = \lambda^3 e^{-4.7\lambda}$$

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

Step 2: Log-likelihood

$$\ell(\lambda) = \ln L(\lambda) = 3\ln\lambda - 4.7\lambda$$

Step 3: First derivative

$$\frac{d\ell}{d\lambda} = \frac{3}{\lambda} - 4.7$$

Set derivative equal to zero:

$$\frac{3}{\hat{\lambda}} - 4.7 = 0 \quad \Rightarrow \quad \hat{\lambda} = \frac{3}{4.7} \approx 0.638$$

Step 4: Second derivative

$$\frac{d^2\ell}{d\lambda^2} = -\frac{3}{\lambda^2} < 0$$

- Negative confirms **maximum**.

MAXIMUM LIKELIHOOD METHOD: EXAMPLE 2

Consider the random variable $X \sim Exp(\lambda)$ and the sample (1.2, 0.5, 3). Determine a **maximum likelihood estimate** of λ.

✅ **Answer**

$$\boxed{\hat{\lambda} \approx 0.638}$$

Shortcut: For an exponential distribution, the MLE of λ is always:

$$\hat{\lambda} = \frac{1}{\bar{X}} = \frac{1}{4.7/3} = \frac{3}{4.7} \approx 0.638$$

Note:

In this case, the **maximum likelihood estimator (MLE) of** λ for the exponential distribution is equal to the **method of moments estimator**:

$$\hat{\lambda}_{\text{MLE}} = \hat{\lambda}_{\text{MM}} = \frac{1}{\bar{X}}$$

However, this **coincidence does not always occur** for other distributions or parameters — in general, the MLE and the method of moments estimator can be different.

INVARIANCE PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATORS

The **invariance property** states that if $\hat{\theta}$ is the **maximum likelihood estimator (MLE)** of a parameter θ, then the MLE of any function of that parameter, $g(\theta)$, is simply the same function applied to the MLE of θ.

In other words, if

$$\hat{\theta}$$

is the MLE of θ, then

$$g(\hat{\theta})$$

is the MLE of $g(\theta)$.

$\dot{\theta}$ *is MLE of* θ → $g(\dot{\theta})$ *is MLE of* $g(\theta)$

Example

If $\hat{\theta}$ is the MLE of θ, and we want to estimate θ^2, then the MLE of θ^2 is

$$\hat{\theta}^2$$

This property simplifies estimation because we do not need to derive a new likelihood function for transformations of the parameter.

INVARIANCE PROPERTY OF MAXIMUM LIKELIHOOD ESTIMATORS: EXAMPLES

Example 1: Exponential distribution

- Let $X_1, \ldots, X_n \sim \text{Exponential}(\lambda)$.
- The MLE of λ is $\hat{\lambda} = 1/\bar{X}$.

Now suppose we want the MLE of the **mean** of the distribution, which is $\mu = 1/\lambda$.

- By invariance:

$$\hat{\mu} = \frac{1}{\hat{\lambda}} = \bar{X}.$$

✅ The MLE of the mean is simply the sample mean.

Example 2: Normal distribution

- Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$.
- The MLE of μ is $\hat{\mu} = \bar{X}$.
- The MLE of σ^2 is $\hat{\sigma}^2 = \frac{1}{n} \sum (X_i - \bar{X})^2$.

Now suppose we want the MLE of the **standard deviation**, $\sigma = \sqrt{\sigma^2}$.

- By invariance:

$$\hat{\sigma} = \sqrt{\hat{\sigma}^2} = \sqrt{\frac{1}{n} \sum (X_i - \bar{X})^2}.$$

✅ The MLE of the standard deviation is the square root of the MLE of the variance.

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COMPARISON BETWEEN THE METHOD OF MOMENTS AND MAXIMUM LIKELIHOOD ESTIMATORS

Method	Method of Moments (MM)	Maximum Likelihood Method (MLM)
Basic idea	Equate sample moments with population moments	Maximize the likelihood function
Computation	Usually simpler	Often more complex
Statistical properties	May not always be efficient	Often efficient and consistent
Use in practice	Useful for quick estimation	Very widely used in statistics
Invariance property	Does not generally satisfy invariance	Satisfies the invariance property

LECTURE 11 HOMEWORK: QUESTIONS AND SOLUTIONS

EXERCISE 1 A), B) AND C)

Let X be a random variable with probability function

$$f(x \mid \theta) = \theta(1-\theta)^x, \quad x = 0, 1, 2, 3, \ldots, \quad 0 < \theta < 1.$$

It is known that $E(X) = \frac{1-\theta}{\theta}$. From a random sample of size $n = 1000$, the following value was observed:

$$\sum_{i=1}^{1000} X_i = 980.$$

a) Obtain an estimate of θ using the **method of moments**.

b) Determine the **maximum likelihood estimator** of θ.

c) Compute, with justification, the **maximum likelihood estimate of the population mean**.

d) Reparametrize the distribution in terms of $\mu = E(X)$, and use the new probability function to estimate the **population mean**.

Murteira (2015), Chapter 7

EXERCISE 1 A): SOLUTION

Answer:

$$f(x|\theta) = \theta(1-\theta)^x \quad (x = 0, 1, 2, 3, \ldots) \text{ com } (0 < \theta < 1)$$

$$E(X) = \frac{1-\theta}{\theta} = \mu_1'$$

Amostra casual: $(X_1, \cdots, X_{1000}) \rightarrow \sum_{i=1}^{1000} x_i = 980$

$$\bar{x} = \frac{980}{1000} = 0.98$$

a) $\theta = ?$

The first population moment of order k : $\mu_1' = E(X) = \frac{1-\theta}{\theta}$

The first sample moment of order k : $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Método dos momentos:

$$\mu_1' = \frac{\sum_{i=1}^{1000} X_i}{1000} \Leftrightarrow \frac{1-\theta}{\theta} = \frac{\sum_{i=1}^{1000} X_i}{1000} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\theta} - 1 = \bar{X} \Leftrightarrow \frac{1}{\theta} = \bar{X} + 1 \Leftrightarrow \theta = \frac{1}{\bar{X}+1}$$

EXERCISE 1 A): SOLUTION

✅ Answer:

Estimator $\theta = \dfrac{1}{\overline{X}+1}$

Estimate: $\tilde{\theta} = \dfrac{1}{\overline{x}+1} = \dfrac{1}{0.98+1} \simeq 0.5051$

Nota:

Population moment of order k

Momento ordinário de ordem k (ou momento de ordem k em relação à origem):

- $\mu'_k = E(X^k)$

Sample Moments of order k

- Momentos ordinários amostrais de ordem k:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^k$$

EXERCISE 1 B): SOLUTION

Answer:

b) $\hat{\theta} = ?$

$$L(\theta) = f(x_1, \dots, x_n | \theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \theta (1-\theta)^{x_i} =$$

$$= \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i} \quad (0<\theta<1)$$

Nota:

$$a^{x_1} \cdot a^{x_2} = a^{x_1+x_2}$$

$$\prod_{i=1}^{n} a^{x_i} = a^{\sum_{i=1}^{n} x_i}$$

$$\ell(\theta) = \ln \{L(\theta)\} =$$

$$= \ln \left\{ \theta^n (1-\theta)^{\sum_{i=1}^{n} x_i} \right\} =$$

$$= n \ln(\theta) + \sum_{i=1}^{n} x_i \ln(1-\theta) \quad (0<\theta<1)$$

$$\frac{d}{d\theta}\{\ell(\theta)\} = 0 \Leftrightarrow \frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta} = 0 \Leftrightarrow$$

$$\Leftrightarrow \frac{n}{\theta} = \frac{\sum_{i=1}^{n} x_i}{1-\theta} \Leftrightarrow \frac{1-\theta}{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{\theta} = \bar{x} + 1 \Leftrightarrow \hat{\theta} = \frac{1}{1+\bar{x}}$$

EXERCISE 1 B): SOLUTION

Answer:

$$\frac{d^2}{d\theta^2}\,\ell(\hat{\theta}) = \frac{d}{d\theta}\left\{\frac{n}{\theta} - \frac{\sum_{i=1}^{n} x_i}{1-\theta}\right\}\Bigg|_{\theta=\hat{\theta}} = -\frac{n}{\theta^2} - \frac{\sum_{i=1}^{n} x_i}{(1-\theta)^2}\Bigg|_{\theta=\hat{\theta}} =$$

$$= \underbrace{-\frac{n}{\hat{\theta}^2}}_{>0} - \underbrace{\frac{\sum_{i=1}^{n} x_i}{(1-\hat{\theta})^2}}_{>0} < 0 \quad \text{para } (0 < \hat{\theta} < 1)$$

↳ Valores admissíveis de θ

porque $n = 1000$ e $\sum_{i=1}^{1000} x_i = 980$

Conclusão: $\hat{\theta} = \frac{1}{1+\bar{X}}$ é o estimador da máxima verosimilhança para θ.

Nota: $\hat{\Theta} = \frac{1}{1+\bar{X}}$ é um estimador (variável aleatória)

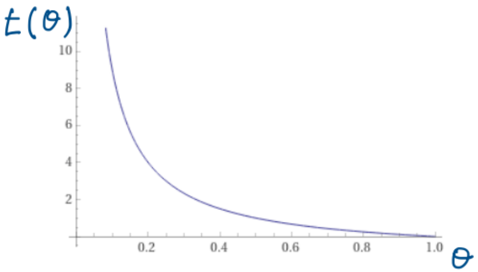
$\hat{\theta} = \frac{1}{1+\bar{x}}$ é uma estimativa ($\hat{\theta} \in \mathbb{R}$)

EXERCISE 1 C): SOLUTION

Answer:

c) $\mu_x = \frac{1-\theta}{\theta} = \frac{1}{\theta} - 1 = t(\theta)$ função de θ

Queremos $\hat{\mu}_x = \widehat{t(\theta)}$ e temos $\hat{\theta}$



$t(\theta)$ é monótona decrescente em $0 < \theta < 1$

⇓

$t(\theta)$ é função biunívoca de θ em $0 < \theta < 1$

⇓

Pode-se usar a propriedade de invariância do EMV

$\hat{\theta} = 0.5051$

$$\hat{\mu}_x = \widehat{t(\theta)} = t(\hat{\theta}) = \frac{1}{\hat{\theta}} - 1 = \frac{1}{0.5051} - 1 = 0.9798$$

Para obter igual às soluções usar $\hat{\theta} = \frac{1}{0.98+1}$ sem arredondar

EXERCISE 5

Consider a random sample of size n drawn from a population with probability density function

$$f(x \mid \theta) = \frac{1}{2\theta}, \quad (-\theta < x < \theta), \quad \text{for } \theta > 0.$$

Compute an estimator for θ using the **method of moments**.

Murteira (2015), Chapter 7

EXERCISE 5: SOLUTION

Answer:

$(X_1, \ldots, X_n)$ Random Sample

$$f_X(x|\theta) = \frac{1}{2\theta} \quad (-\theta < x < \theta), \quad \theta > 0$$

Distribuição uniforme contínua

$$E(X) = \int_{-\theta}^{\theta} x \cdot f_X(x|\theta)\, dx =$$

$$= \int_{-\theta}^{\theta} x \cdot \frac{1}{2\theta}\, dx = \frac{1}{2\theta}\left[\frac{x^2}{2}\right]_{-\theta}^{\theta} =$$

$$= \frac{1}{2\theta}\left(\frac{\theta^2}{2} - \left(\frac{(-\theta)^2}{2}\right)\right) = \frac{1}{2\theta}\left(\frac{\theta^2}{2} - \frac{\theta^2}{2}\right) = 0$$

Note: Since E(X) does not depend on Θ, we will try the second moment

.

EXERCISE 5: SOLUTION

Answer:

$$E(X^2) = \int_{-\theta}^{\theta} x^2 f_X(x|\theta)\,dx =$$

$$= \int_{-\theta}^{\theta} x^2 \cdot \frac{1}{2\theta}\,dx = \frac{1}{2\theta}\left[\frac{x^3}{3}\right]_{-\theta}^{\theta} =$$

$$= \frac{1}{2\theta}\left(\frac{\theta^3}{3} - \left(\frac{(-\theta)^3}{3}\right)\right) =$$

$$= \frac{1}{2\theta}\left(\frac{\theta^3}{3} + \frac{\theta^3}{3}\right) = \frac{1}{2\theta}\left(2\,\frac{\theta^3}{3}\right) = \frac{\theta^2}{3}$$

EXERCISE 5: SOLUTION

Answer:

M.M:

$$E(X^2) = \sum_{i=1}^{n} X_i^2 / n \Leftrightarrow$$

$$\Leftrightarrow \frac{\theta^2}{3} = \sum_{i=1}^{n} X_i^2 / n \Leftrightarrow$$

$$\Leftrightarrow \theta^2 = \frac{3}{n} \sum_{i=1}^{n} X_i^2 \Leftrightarrow \theta = \pm\sqrt{\frac{3}{n} \sum_{i=1}^{n} X_i^2}$$

$$\Rightarrow \tilde{\theta} = +\sqrt{\frac{3}{n} \sum_{i=1}^{n} X_i^2} \quad (\text{porque } \theta > 0)$$

EXERCISE 5: SOLUTION

Answer:

Nota: Caso se reparasse que $X \sim U(-\theta, \theta)$, então:

$$E(X) = \frac{-\theta + \theta}{2} = 0$$

$$E(X^2) = Var(X) + \underbrace{E(X)}_{0}{}^2 = \frac{(\theta - (-\theta))^2}{12} = \frac{(2\theta)^2}{12} = \frac{\theta^2}{3}$$

EXERCISE 1: MLM

Let $X_1, \ldots, X_n$ be a random sample from a Normal distribution with parameters μ and σ. Estimate the parameters using the **maximum likelihood method**.

EXERCISE 1 - MLM: SOLUTION

b) Função densidade de probabilidade (f. d. p.):

$$f(x;\ \mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < \mu < +\infty, \qquad \sigma > 0.$$

Função de verosimilhança:

$$\mathcal{L}(\mu,\sigma^2) = \prod_{i=1}^{n} f(x_i\,;\mu\,,\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma^2}\right)^2} = \frac{1}{\left(\sqrt{2\pi\sigma^2}\right)^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}.$$

Logaritmo da função de verosimilhança:

$$\ln\left(\mathcal{L}(\mu,\sigma^2)\right) = -\frac{n}{2}\left(\ln(2) + \ln(\pi) + \ln(\sigma^2)\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2.$$

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EXERCISE 1 - MLM: SOLUTION

Condições de 1ª ordem:

$$\begin{cases} \dfrac{\partial \ln L\,(\mu,\sigma^2)}{\partial \mu} = 0 \\ \dfrac{\partial \ln L\,(\mu,\sigma^2)}{\partial \sigma^2} = 0 \end{cases} \Leftrightarrow \begin{cases} -\dfrac{1}{2\sigma^2}\left(-2\displaystyle\sum_{i=1}^{n} x_i + 2n\mu\right) = 0 \\ -\dfrac{n}{2}\dfrac{1}{\sigma^2} + \displaystyle\sum_{i=1}^{n}(x_i-\mu)^2\,\dfrac{2}{4\sigma^4} = 0 \end{cases} \Leftrightarrow \begin{cases} \displaystyle\sum_{i=1}^{n} x_i - n\mu = 0 \\ -n\sigma^2 + \displaystyle\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \mu = \displaystyle\sum_{i=1}^{n}\dfrac{x_i}{n} \\ \sigma^2 = \displaystyle\sum_{i=1}^{n}\dfrac{(x_i-\mu)^2}{n} \end{cases} \Leftrightarrow \begin{cases} \mu = \overline{x} \\ \sigma^2 = \displaystyle\sum_{i=1}^{n}\dfrac{(x_i-\overline{x})}{n} \end{cases} \Leftrightarrow \begin{cases} \mu = \overline{x} \\ \sigma^2 = \left(\dfrac{n-1}{n}\right)s^2 \end{cases}$$

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EXERCISE 1 - MLM: SOLUTION

Condições de 2ª ordem:

$$\begin{cases} \dfrac{\partial^2 \ln L(\mu, \sigma^2)}{\partial \mu^2} = -\dfrac{1}{2\sigma^2} 2n < 0 \\ \dfrac{\partial^2 \ln L(\mu, \sigma^2)}{\partial \sigma^4} = \dfrac{n}{2}\dfrac{1}{\sigma^4} - \displaystyle\sum_{i=1}^{n} \dfrac{(x_i - \mu)^2}{\sigma^6} < 0 \end{cases},$$

pois $n > 0, \sigma^2 > 0, \sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^6} > \frac{n}{2}\frac{1}{\sigma^4}$.

Portanto, os estimadores de máxima verosimilhança obtidos foram:

$$\begin{cases} \hat{\mu} = \overline{X} = \displaystyle\sum_{i=1}^{n} \dfrac{X_i}{n} \\ \hat{\sigma}^2 = \displaystyle\sum_{i=1}^{n} \dfrac{\left(X_i - \overline{X}\right)^2}{n} = \dfrac{n-1}{n} S^2 \end{cases}.$$

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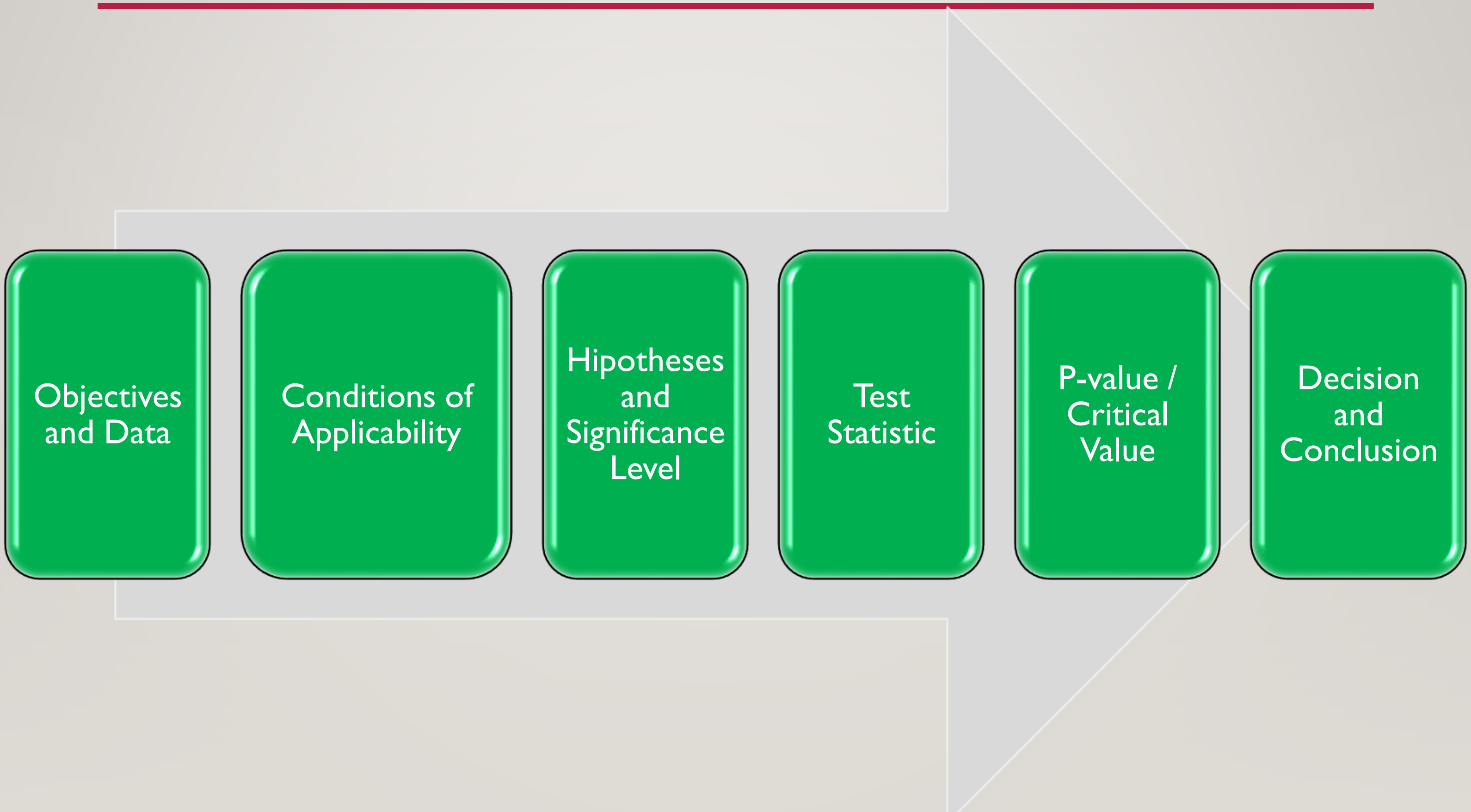
LECTURE 13: HYPOTHESIS TESTS

CONSTRUCTION OF A HYPOTHESIS TEST



WHAT IS A HYPOTHESIS TEST?

- A **hypothesis test** is a statistical procedure used to make a decision or draw a conclusion about a population parameter, based on sample data.
  It evaluates whether there is enough evidence to reject a claim (the **null hypothesis**) in favor of an alternative claim (the **alternative hypothesis**), using a test statistic.

A hypothesis is an assumption about the population parameter (say population mean) which is to be tested.

For that we collect sample data , then we calculate sample statistics (say sample mean) and then use this information to judge/decide whether hypothesized value of population parameter is correct or not.

WHAT IS A HYPOTHESIS?

Note:
Different null and alternative hypotheses can be considered depending on the parameter of interest, and the **type of test** is determined accordingly.

A hypothesis is a statement about a population parameter that can be tested using sample data.

Example: The mean weight of this class is 58 kg?

Note: To test this question, we use a **test for a population mean**, since the **parameter of interest is the mean**.

Null and Alternative Hypotheses:

- The **null hypothesis (H_0)** is a statement of no effect or no difference. It represents the default or status quo.
- The **alternative hypothesis (H_1 or H_a)** is a statement that contradicts the null, representing the effect or difference we want to detect.

Note: There are **three types of tests for the mean**, depending on the objective of the test: **Two-tailed test, Right-tailed test and Left-tailed test**

Examples for the population mean (μ):

Type of test	Null hypothesis (H_0)	Alternative hypothesis (H_1)	
Two-tailed	$\mu = \mu_0$	$\mu \neq \mu_0$	$H_0: \mu = 58$ vs $H_1: \mu \neq 58$
Right-tailed	$\mu \leq \mu_0$	$\mu > \mu_0$	$H_0: \mu \leq 58$ vs $H_1: \mu > 58$
Left-tailed	$\mu \geq \mu_0$	$\mu < \mu_0$	$H_0: \mu \geq 58$ vs $H_1: \mu < 58$

CONCEPTS OF HYPOTHESIS TESTING

- A hypothesis is a claim (assumption) about a population parameter:
  - population mean

    Example: The mean monthly cell phone bill of this city is $\mu = \$52$

    Hypothesis Test for a Population Mean
  - population proportion

    Example: The proportion of adults in this city with cell phones is $P = .88$

    Hypothesis Test for a Population Proportion

Note: The type of hypothesis test depends on the **parameter of interest**:
Mean → hypothesis test for a population mean
Proportion → hypothesis test for a population proportion.

Newbold et al (2013)

THE NULL HYPOTHESIS, H SUB 0

- States the assumption (numerical) to be tested

  Example: The average number of TV sets in U.S. Homes is equal to three $(H_0 : \mu = 3)$

  H_0: μ = 3 vs H_1: μ ≠ 3

- Is always about a population parameter, not about a sample statistic

$$H_0 : \mu = 3$$

~~$H_0 : \bar{x} = 3$~~

Newbold et al (2013)

THE NULL HYPOTHESIS, H SUB 0

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- Refers to the status quo
- Always contains "=", "≤" or "≥" sign
- May or may not be rejected

Examples:

$H_0: \mu = 3$ vs $H_1: \mu \neq 3$

$H_0: \mu \leq 3$ vs $H_1: \mu > 3$

$H_0: \mu \geq 3$ vs $H_1: \mu < 3$

THE ALTERNATIVE HYPOTHESIS, H SUB 1

- Is the opposite of the null hypothesis
  - e.g., The average number of TV sets in U.S. homes is not equal to 3 $(H_1 : \mu \neq 3)$
- Challenges the status quo
- Never contains the "=", "≤" or "≥" sign
- May or may not be supported
- Is generally the hypothesis that the researcher is trying to support

Examples:

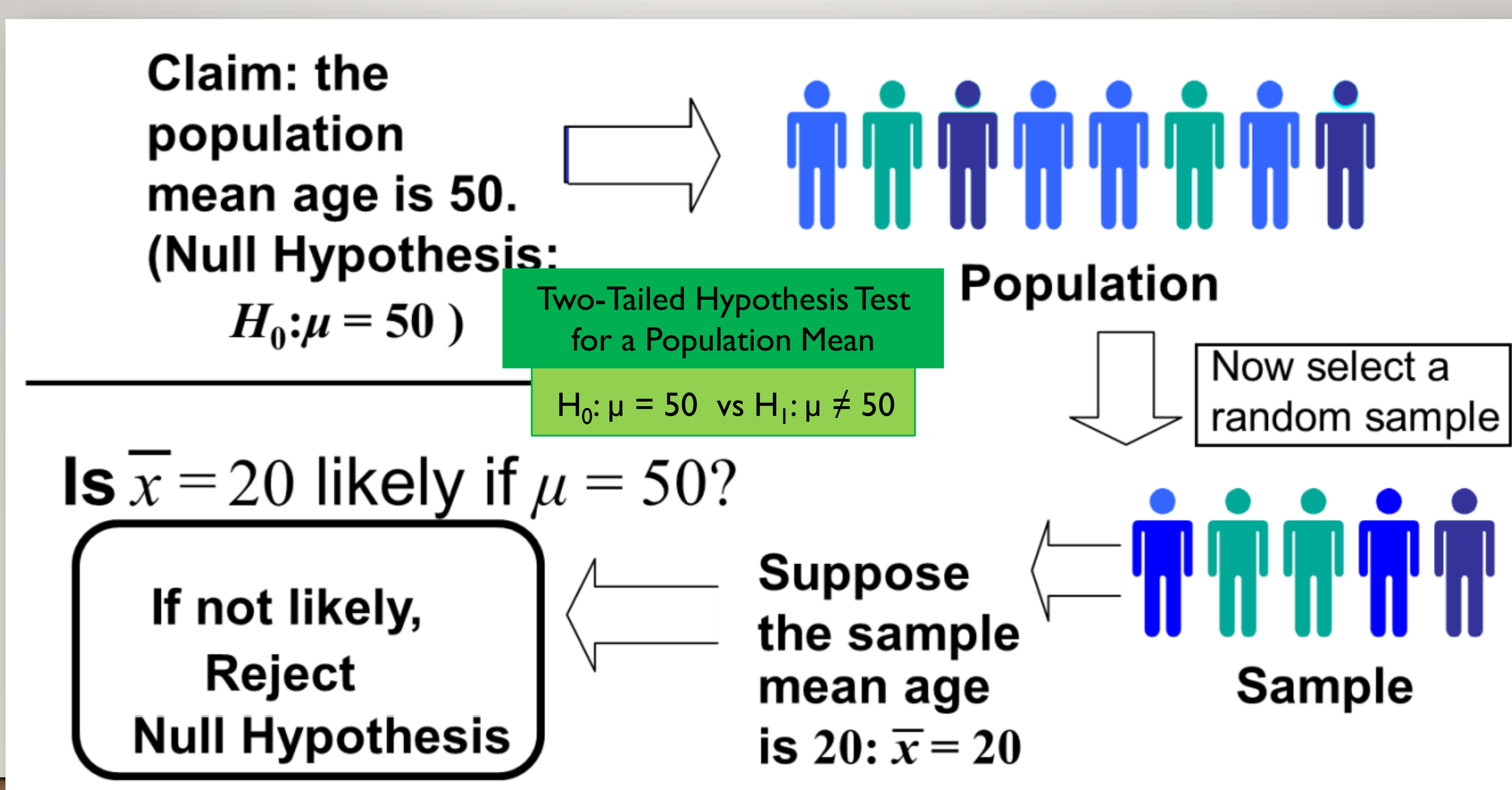
$H_0: \mu = 3$ vs $H_1: \mu \neq 3$

$H_0: \mu \leq 3$ vs $H_1: \mu > 3$

$H_0: \mu \geq 3$ vs $H_1: \mu < 3$

Newbold et al (2013)

HYPOTHESIS TESTING PROCESS



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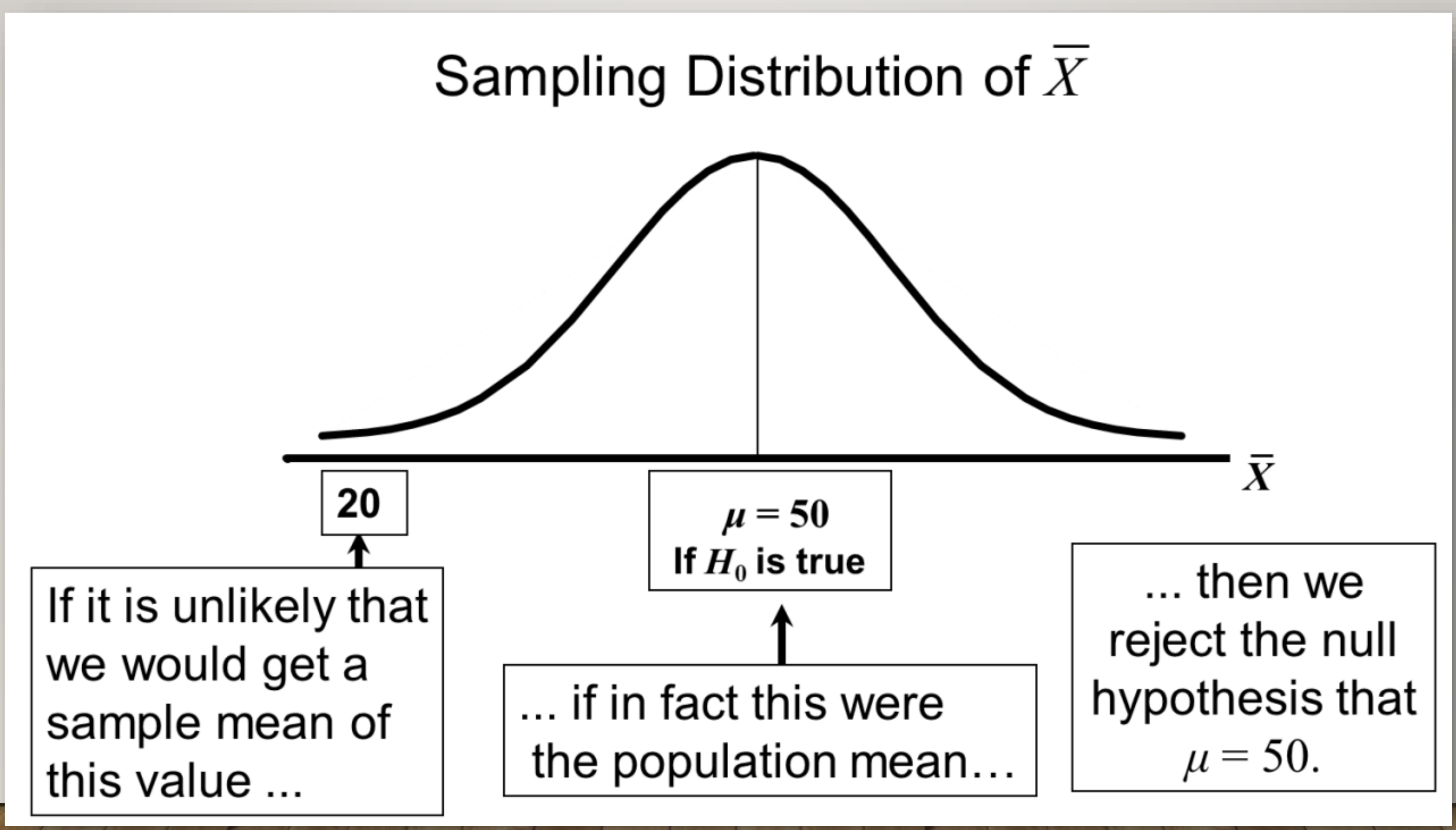
REASON FOR REJECTING H SUB 0



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LEVEL OF SIGNIFICANCE

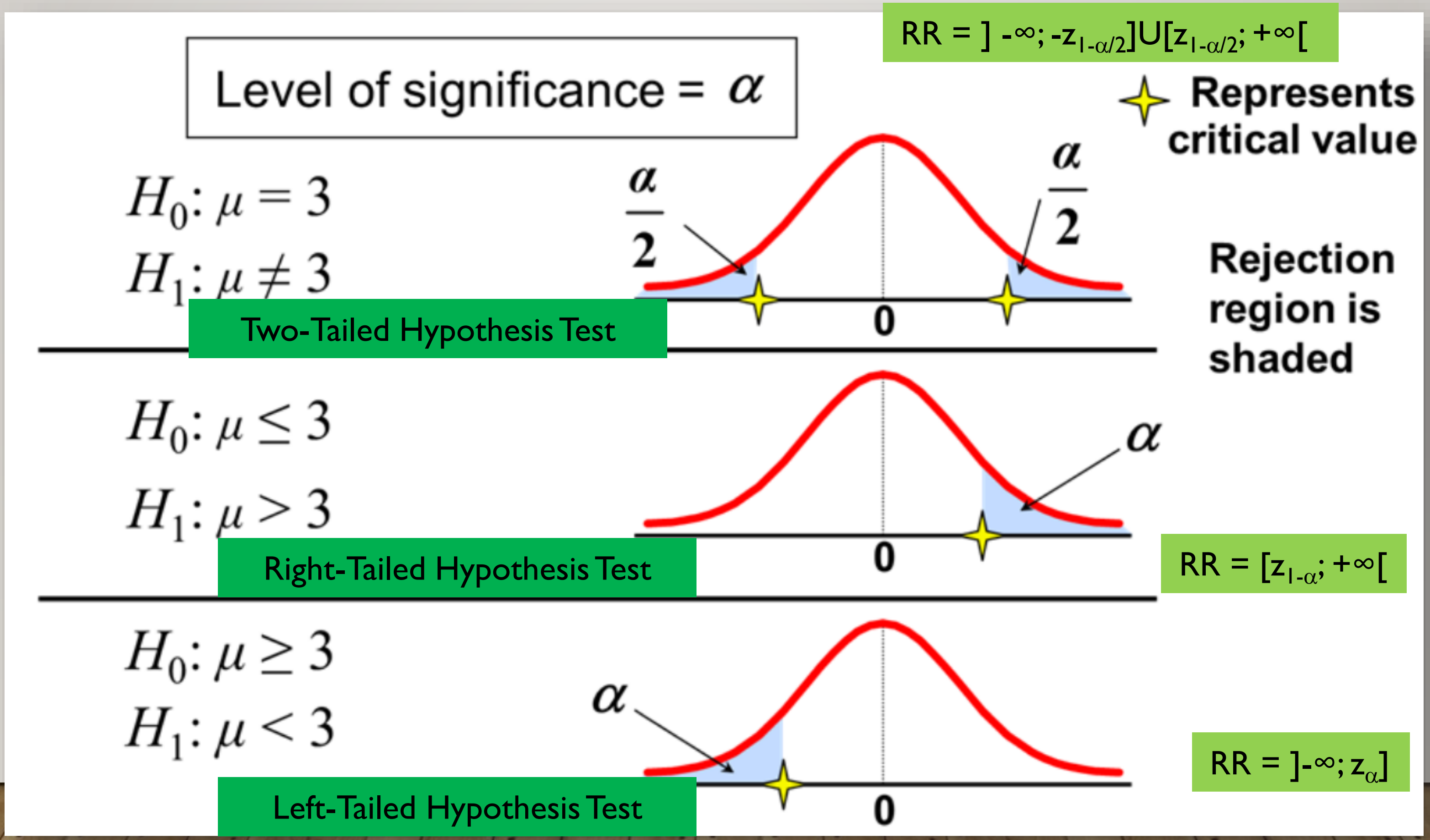
Type I and Type II Errors

- α = P(Type I error) = $P(\text{reject } H_0 \mid H_0 \text{ is true})$
- β = P(Type II error) = $P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by ***α***, (level of significance)
  - Typical values are 0.01, 0.05, or 0.10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test

Newbold et al (2013)

LEVEL OF SIGNIFICANCE AND THE REJECTION REGION (RR)



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ERRORS IN MAKING DECISIONS

Type I and Type II Errors

- α = P(Type I error) = $P(\text{reject } H_0 \mid H_0 \text{ is true})$
- β = P(Type II error) = $P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$

- **Type I Error**
  - Reject a true null hypothesis
  - Considered a serious type of error

The probability of Type I Error is $\boldsymbol{\alpha}$

- Called level of significance of the test
- Set by researcher in advance

- **Type II Error**
  - Fail to reject a false null hypothesis

The probability of Type II Error is β

Newbold et al (2013)

TYPES OF ERRORS AND SIGNIFICANCE LEVEL

Type I and Type II Errors

- α = P(Type I error) = $P(\text{reject } H_0 \mid H_0 \text{ is true})$
- β = P(Type II error) = $P(\text{fail to reject } H_0 \mid H_0 \text{ is false})$

Types of Errors

Type I error (α): Rejecting H_0 when H_0 is true.
Type II error (β): Failing to reject H_0 when H_0 is false.

Power of a Test

Condition	Reject H_0	Accept H_0
H_0 is false	Correct decision 1-β	Incorrect decision (Typo II error) β
H_0 is true	Incorrect decision (Type I error) α	Correct decision 1-α

Power of a Test:
The power of a test is the probability of correctly rejecting the null hypothesis (H_0) when it is false.
Mathematically, it is given by:
Power $= 1 - \beta$
where β is the probability of a Type II error.

Significance Level ($0 \le \alpha \le 1$)

- The probability that the researcher sets a priori as the threshold to decide whether to reject H_0.
- Common significance levels: 1%, 5%, and 10%

OUTCOMES AND PROBABILITIES

Possible Hypothesis Test Outcomes

	Actual Situation	
Decision	H_0 True	H_0 False
Fail to Reject H_0	**Correct Decision** $(1-\alpha)$	**Type II Error** (β)
Reject H_0	**Type I Error** (α)	**Correct Decision** $(1-\beta)$

Key:
Outcome
(Probability)

$(1-\beta)$ is called the power of the test

Newbold et al (2013)

TYPE I & II ERROR RELATIONSHIP

- Type I and Type II errors can not happen at the same time
  - Type I error can only occur if H_0 is true
  - Type II error can only occur if H_0 is false

If Type I error probability $(\alpha) \Uparrow$, then

Type II error probability $(\beta) \Downarrow$

Newbold et al (2013)

FACTORS AFFECTING TYPE II ERROR

- All else equal,
  - $\beta \Uparrow$ when the difference between hypothesized parameter and its true value $\Downarrow$
  - $\beta \Uparrow$ when $\alpha \Downarrow$
  - $\beta \Uparrow$ when $\sigma \Uparrow$
  - $\beta \Uparrow$ when $n \Downarrow$

Newbold et al (2013)

POWER OF THE TEST

- The power of a test is the probability of rejecting a null hypothesis that is false
- i.e., $\text{Power} = P(\text{Reject } H_0 \mid H_1 \text{ is true})$
  - Power of the test increases as the sample size increases

Newbold et al (2013)

REJECTION REGION VS. P-VALUE

Rejection Region

- The rejection region is also called the critical region , is the range of sample statistics values within which if values of sample statistics falls , then Ho rejected.
- It is outside the limit of acceptance region.
- The critical value is the cut off value of the sample statistics which acts as a boundary and separates the regions of acceptance or rejections

Rejection Region (RR) or Critical Region (CR): The set of values for which H_0 is rejected

- Left-Tailed Test: RR = $]-\infty; z_{\alpha}]$
- Right-Tailed Test: RR = $[z_{1-\alpha}; +\infty[$
- Two-Tailed Test : RR = $] -\infty; -z_{1-\alpha/2}] \cup [z_{1-\alpha/2}; +\infty[$

"Decision Rule (using critical values):

- $z_0 \leq z_\alpha \Rightarrow$ Reject H_0
- $z_0 \geq z_{1-\alpha} \Rightarrow$ Reject H_0
- $|z_0| \geq z_{1-\alpha/2} \Rightarrow$ Reject H_0"

"Rule: $z_0 \in \mathrm{RR} \Rightarrow$ Reject H_0"

Note:
If the value of the test statistic falls within the rejection region, then we reject H_0 at the chosen significance level.

P-value: is the probability of obtaining a test statistic at least as extreme as the one observed, assuming that hull hypotesis H_0 is true.

- Left-Tailed Test: P-value = $P(Z \leq z_0)$
- Right-Tailed Test: P-value = $P(Z \geq z_0)$
- Two-Tailed Test: P-value = $P(Z \leq -z_0 \text{ or } Z \geq z_0) = 2 \times P(Z \geq |z_0|)$

"Rule: P-value < $\alpha \Rightarrow$ Reject H_0"

Note:
The **P-value** helps to determine whether the observed data are consistent with H_0:

- A **small p-value** (typically $\leq \alpha$) indicates strong evidence against H_0, so we reject H_0.
- A **large p-value** ($> \alpha$) indicates weak evidence against H_0, so we fail to reject H_0.

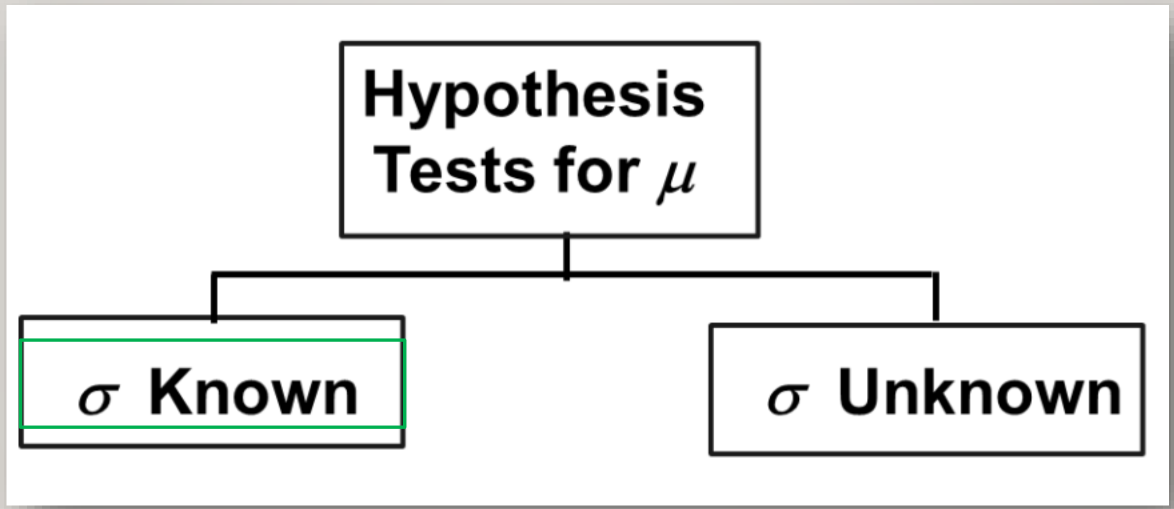
Note:
If the p-value is smaller than the chosen significance level (α), then we reject H_0 at that significance level.

REJECTION REGION VS. P-VALUE

- ✓ The decision of a statistical test can be made **either using the rejection region or the p-value**. It is sufficient to use just one of these methods because **both approaches always lead to the same conclusion**.
- ✓ The **p-value does not depend on the chosen significance level** (α); it is a property of the observed data. In contrast, the **rejection region depends on the significance level** and may vary with different α values.

LECTURE 13: TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN)

HYPOTHESIS TESTS FOR THE MEAN



Newbold et al (2013)

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN)

1. Hypotheses

- Null hypothesis: $H_0 : \mu = \mu_0$, H_0: $\mu \leq \mu_0$ OR H_0: $\mu \geq \mu_0$
- Alternative hypothesis: $H_1 : \mu \neq \mu_0$ (two-tailed)
  or $H_1 : \mu > \mu_0$ / $H_1 : \mu < \mu_0$ (one-tailed)

Note: For a hypothesis test on the population mean with **known variance**, one assumption is that the **population distribution is normal** (particularly important for small sample sizes).

2. Test Statistic

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim \text{Normal}(0, 1)$$

where $\bar{X}$ is the sample mean, σ the population standard deviation, and n the sample size.

3. Decision Rule

- Using critical value(s): Reject H_0 if $Z \in \text{Rejection Region}$
- Using p-value: Reject H_0 if $p\text{-value} < \alpha$

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN)

- A parametric hypothesis test for the parameter μ (the population mean) may be:

Two-Tailed Test

$H_0: \mu = \mu_0$

$H_1: \mu \neq \mu_0$

RR = $]-\infty; -z_{1-\alpha/2}] \cup [z_{1-\alpha/2}; +\infty[$

Right-Tailed Test

$H_0: \mu \leq \mu_0$

$H_1: \mu > \mu_0$

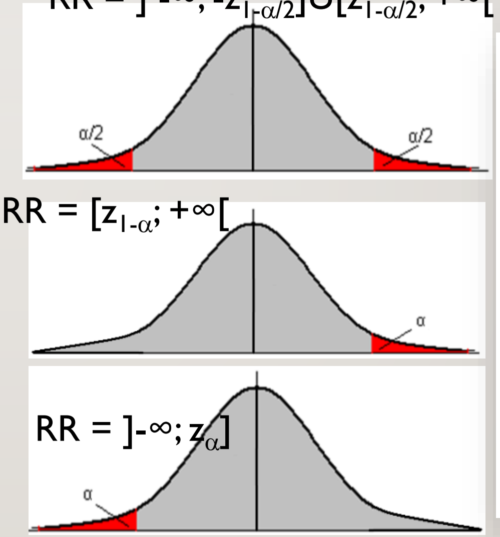
RR = $[z_{1-\alpha}; +\infty[$

Left-Tailed Test

$H_0: \mu \geq \mu_0$

$H_1: \mu < \mu_0$

RR = $]-\infty; z_{\alpha}]$



Two-tailed test:

- Rejection Region: $|Z| \geq Z_{1-\alpha/2}$
- P-value: $2 \cdot P(Z \geq |z_0|)$

Right-tailed test:

- Rejection Region: $Z \geq Z_{1-\alpha}$
- P-value: $P(Z \geq z_0)$

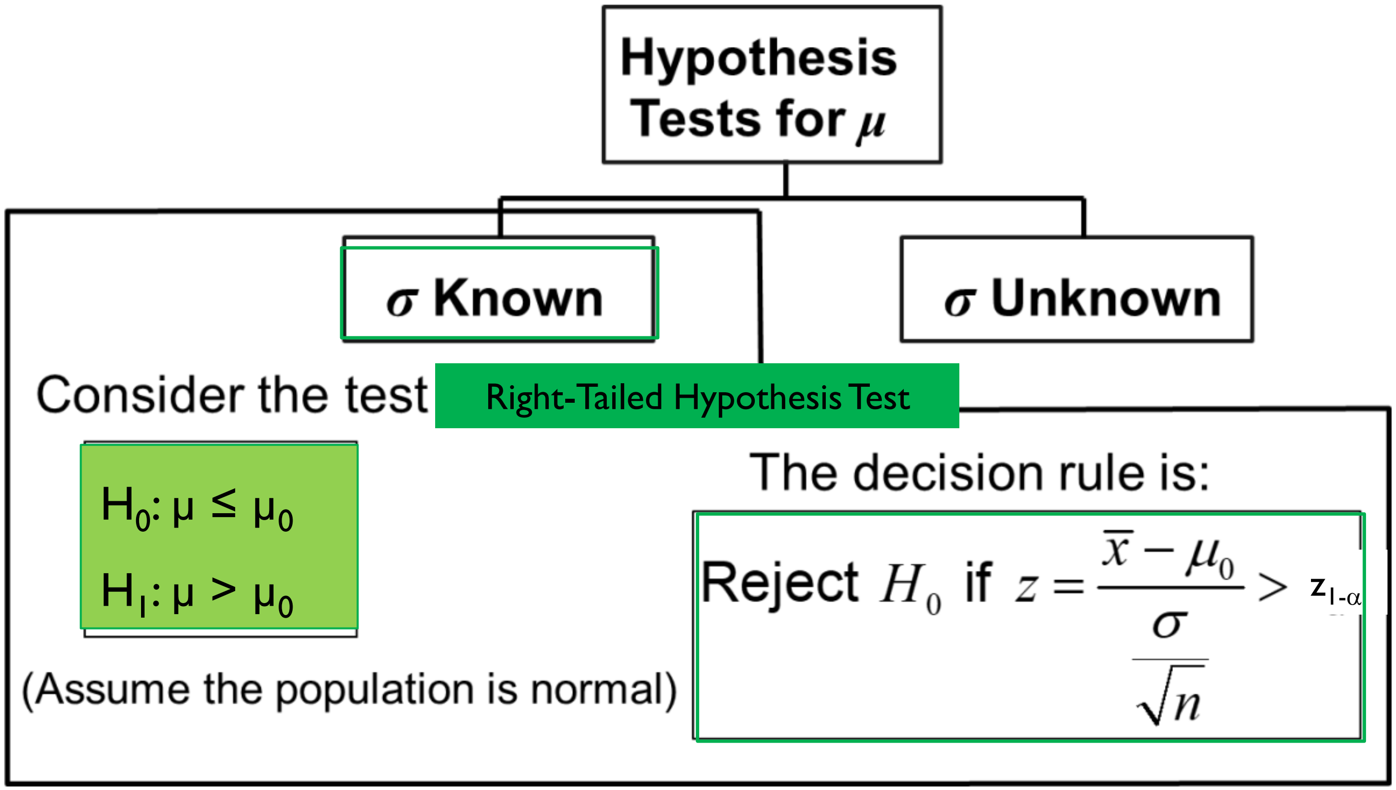
Left-tailed test:

- Rejection Region: $Z \leq Z_{\alpha}$
- P-value: $P(Z \leq z_0)$

where μ_0 is the specific numerical value considered in H_0 and H_1.

TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN): EXAMPLE

- Convert sample result $(\bar{x})$ to a z value



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TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN): EXAMPLE

Decision Rule

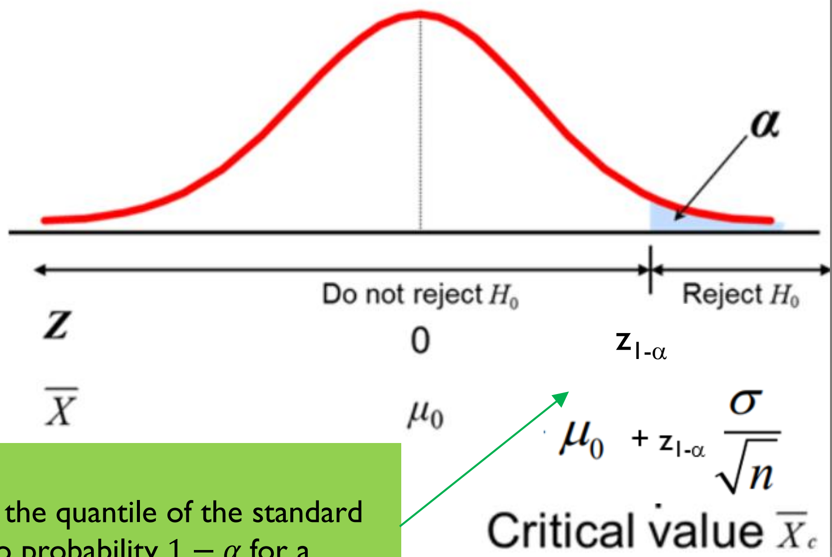
Reject H_0 if $z = \dfrac{\bar{x} - \mu_0}{\dfrac{\sigma}{\sqrt{n}}} > z_{1-\alpha}$

Alternate rule:

Reject H_0 if $\bar{x} > \mu_0 + z_{1-\alpha} \dfrac{\sigma}{\sqrt{n}}$

H_0: $\mu \leq \mu_0$

H_1: $\mu > \mu_0$



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Note:
The critical value can be obtained as the quantile of the standard normal distribution, corresponding to probability $1-\alpha$ for a right-tailed test: $z_{1-\alpha}$.
Alternatively, for the original scale, it can be expressed as:

$$\text{Critical } value = \mu_0 + Z_{1-\alpha} \cdot \frac{\sigma}{\sqrt{n}}$$

P-VALUE

- *p*-value: Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - Smallest value of α for which H_0 can be rejected

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TESTS OF THE MEAN OF A NORMAL DISTRIBUTION (σ^2 KNOWN): EXAMPLE

P-value

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., ***z*** statistic)
- Obtain the *p*-value
  - For an [Right-Tailed Hypothesis Test]

Right-Tailed Hypothesis Test

$$p\text{-value} = P\left(z > \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}}, \text{ given thet } H_0 \text{ is true}\right)$$

$$= P\left(z > \frac{\bar{x}-\mu_0}{\frac{\sigma}{\sqrt{n}}} \middle| \mu = \mu_0\right)$$

- Decision rule: compare the *p*-value to ***α***
  - If *p*-value $< \alpha$, reject H_0
  - If *p*-value $\geq \alpha$, do not reject H_0

Newbold et al (2013)

RIGHT-TAILED Z TEST FOR MEAN SIGMA KNOWN: EXAMPLE 1

A phone industry manager thinks that customer monthly cell phone bill have increased, and now average over $52 per month. The company wishes to test this claim. (Assume $\sigma = 10$ is known)

Form hypothesis test: **Right-Tailed Hypothesis Test**

$H_0 : \mu \leq 52$ the average is not over $52 per month

$H_1 : \mu > 52$ the average is greater than $52 per month (i.e., sufficient evidence exists to support the manager's claim)

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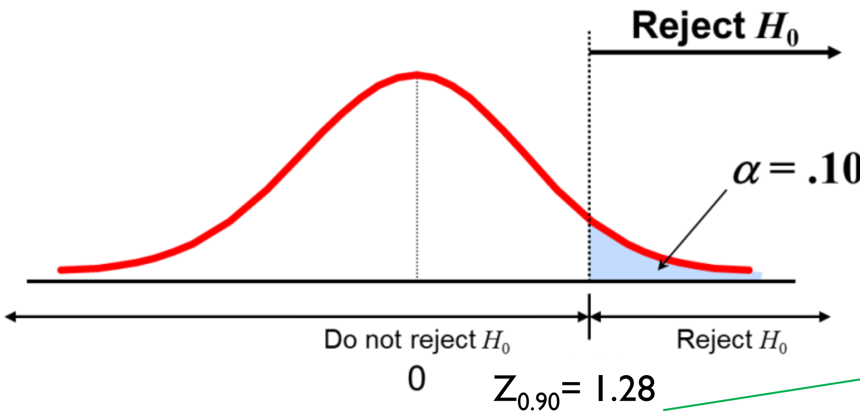
FIND REJECTION REGION (RR): EXAMPLE 1

- Suppose that $\alpha = .10$ is chosen for this test

Find the rejection region:



RR = $[z_{1-\alpha}; +\infty[$

RR = $[1.28; +\infty[$

"$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} \in \text{RR} \Rightarrow$ Reject H_0" $\Leftrightarrow$ Reject H_0 if $z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > 1.28$

Newbold et al (2013)

THE VALUE OF THE TEST STATISTIC: EXAMPLE 1

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$ ($\sigma = 10$ was assumed known)

- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Note:
The value of the test statistic is calculated using the sample data:

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

where $\bar{X}$ is the sample mean, n is the sample size, and σ is the known population standard deviation.

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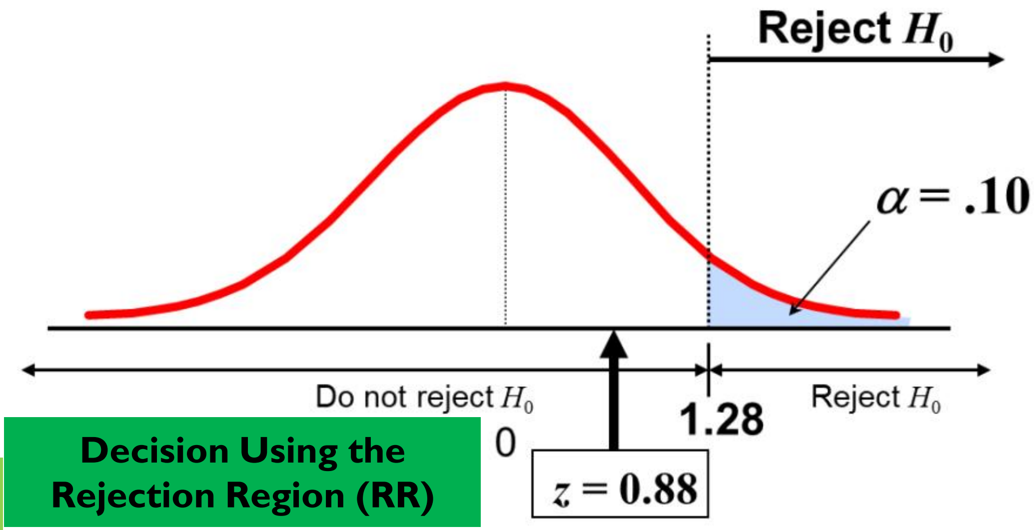
DECISION USING THE RR: EXAMPLE 1

Reach a decision and interpret the result:



Decision Using the Rejection Region (RR)

Do not reject H_0 since $z = 0.88 < 1.28$

i.e.: there is not sufficient evidence that the mean bill is over \$52

Note (Right-Tailed Test): If the value of the test statistic does **not** fall in the rejection region, this means that the test statistic is **less than the critical value**. In this case, we **fail to reject** H_0.

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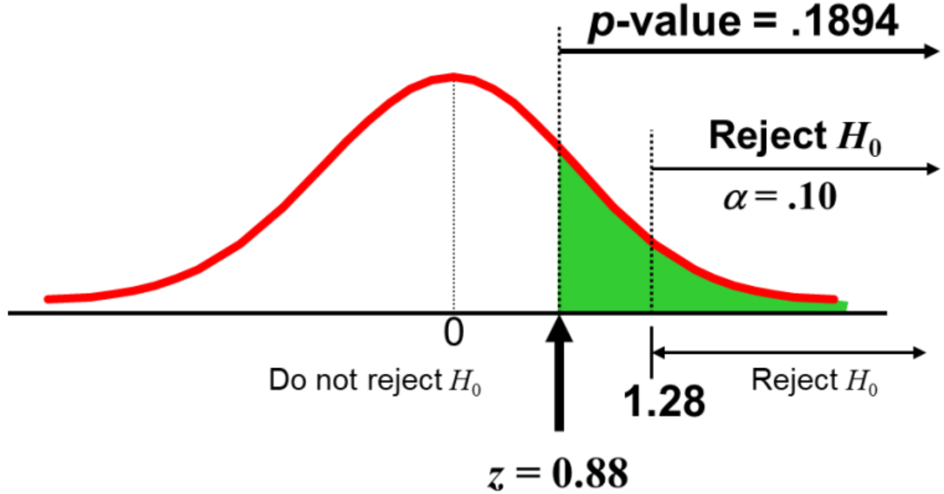
DECISION USING P-VALUE: EXAMPLE 1

Calculate the *p*-value and compare to ***α***

(assuming that $\mu = 52.0$)



$$P(\bar{x} \geq 53.1 \mid \mu = 52.0)$$

$$= P\left(z \geq \frac{53.1 - 52.0}{\frac{10}{\sqrt{64}}} \right)$$

$$= P(z \geq 0.88) = 1 - .8106$$

$$= .1894$$

Do not reject H_0 since *p*-value $= .1894 > \alpha = .10$

"Rule: P-value < α ⇒ Reject H_0"

ONE-TAIL TESTS

- In many cases, the alternative hypothesis focuses on one particular direction

Right-Tailed Hypothesis Test

$$H_0 : \mu \leq 3$$
$$H_1 : \mu > 3$$

⇨ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

Left-Tailed Hypothesis Test

$$H_0 : \mu \geq 3$$
$$H_1 : \mu < 3$$

⇨ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

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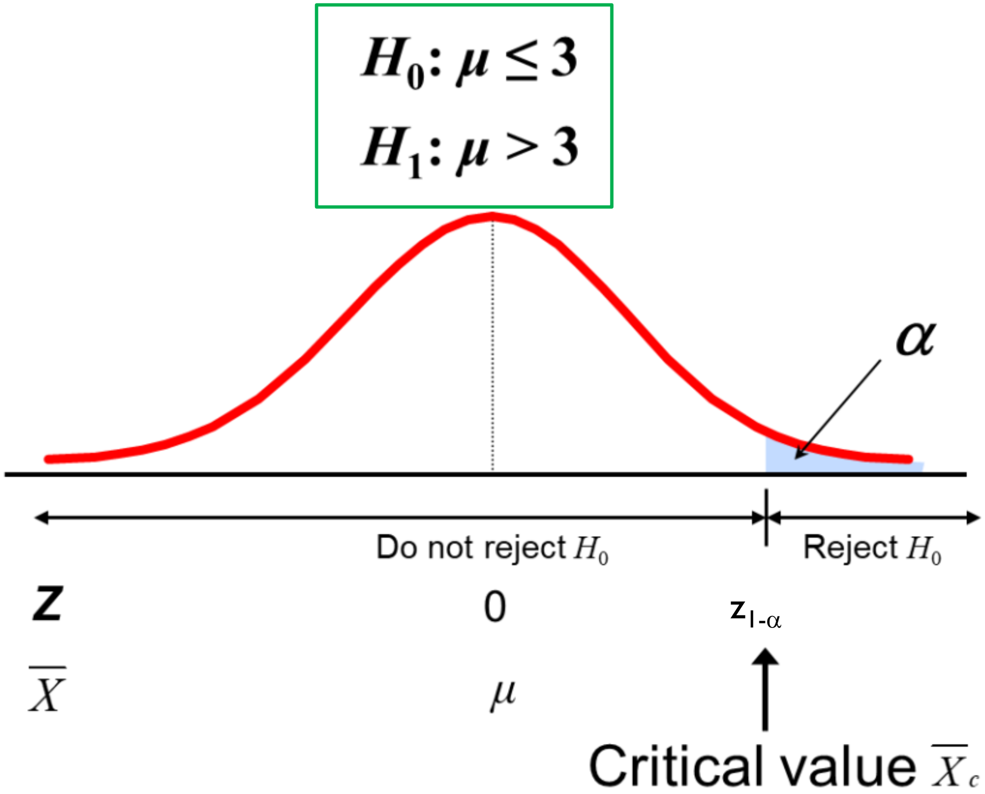
RIGHT-TAILED TESTS

- There is only one critical value, since the rejection area is in only one tail



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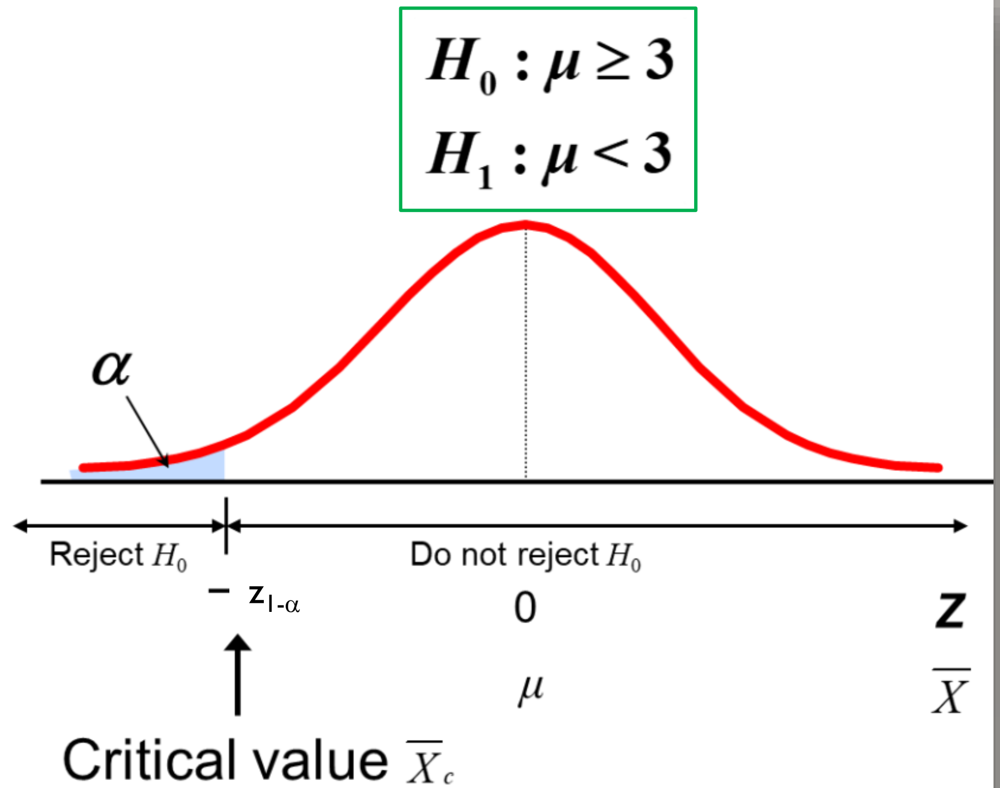
LEFT-TAILED TESTS

- There is only one critical value, since the rejection area is in only one tail



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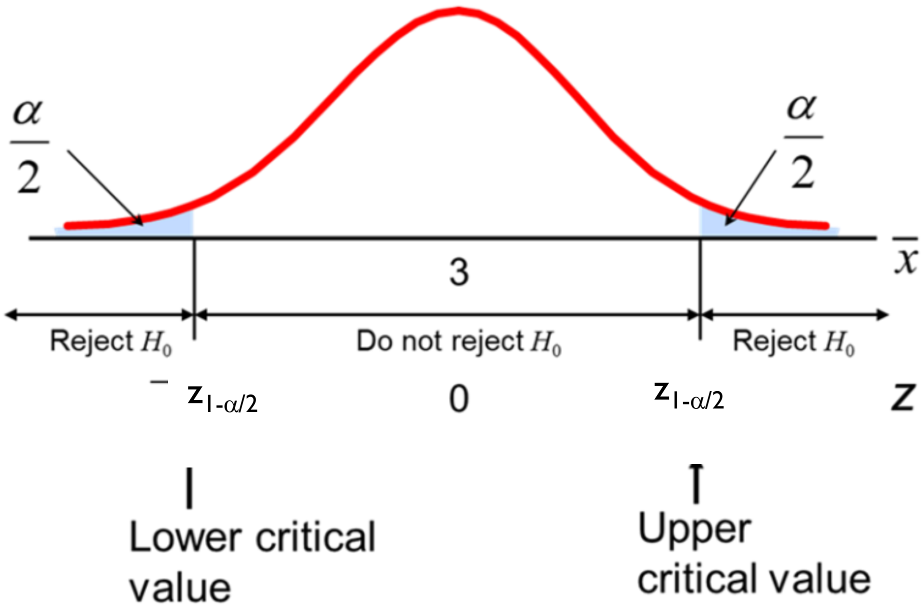
TWO-TAILED TESTS

- In some settings, the alternative hypothesis does not specify a unique direction

- There are two critical values, defining the two regions of rejection

$$H_0 : \mu = 3$$
$$H_1 : \mu \neq 3$$



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HYPOTHESIS TESTING: EXAMPLE 2

Test the claim that the true mean # of TV sets in US homes is equal to 3. (Assume $\sigma = 0.8$)

- State the appropriate null and alternative hypotheses
  - $H_0 : \mu = 3,\ H_1 : \mu \neq 3$ (This is a Two-Tailed Hypothesis Test
- Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test
- Choose a sample size
  - Suppose a sample of size n = 100 is selected

Newbold et al (2013)

HYPOTHESIS TESTING: EXAMPLE 2

- Determine the appropriate technique
  - σ is known so this is a *z* test
- Set up the critical values
  - For $\alpha = .05$ the critical *z* values are ± 1.96

$RR =]-\infty; -z_{1-\alpha/2}] \cup [z_{1-\alpha/2}; +\infty[$

$RR =]-\infty; -1.96] \cup [1.96; +\infty[$

- Collect the data and compute the test statistic
  - Suppose the sample results are
    $n = 100,\ \bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

  So the test statistic is:

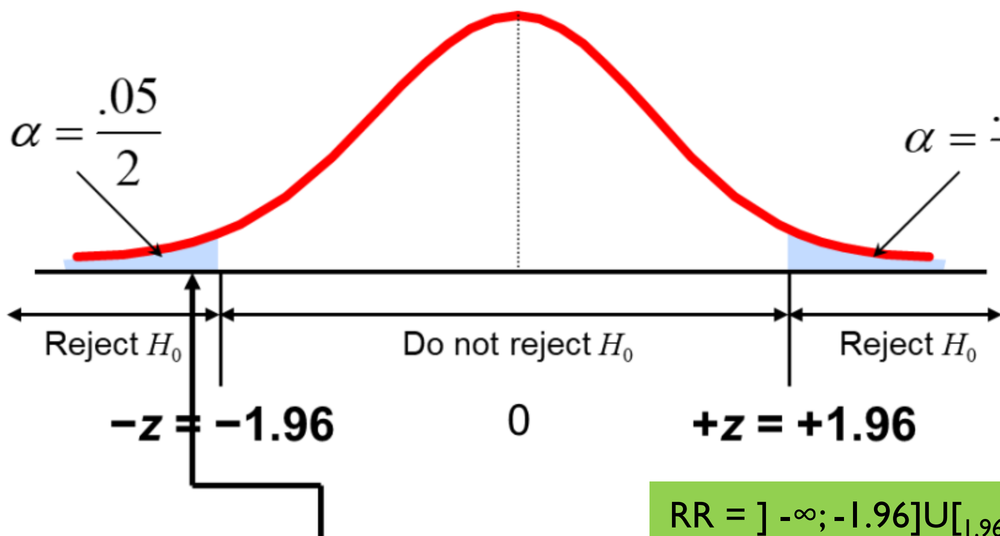
$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Newbold et al (2013)

HYPOTHESIS TESTING: EXAMPLE 2

- Is the test statistic in the rejection region?

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0



Note: The test statistic $Z_0 = -2.0$ belongs to the rejection region, so we reject H_0 at a 5% significance level.

RR =] -∞; -1.96]U[1.96; +∞[

Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

$Z_0 = -2.0 \in \text{Rejection Region (RR)} \implies \text{Reject } H_0 \text{ at } \alpha = 0.05$

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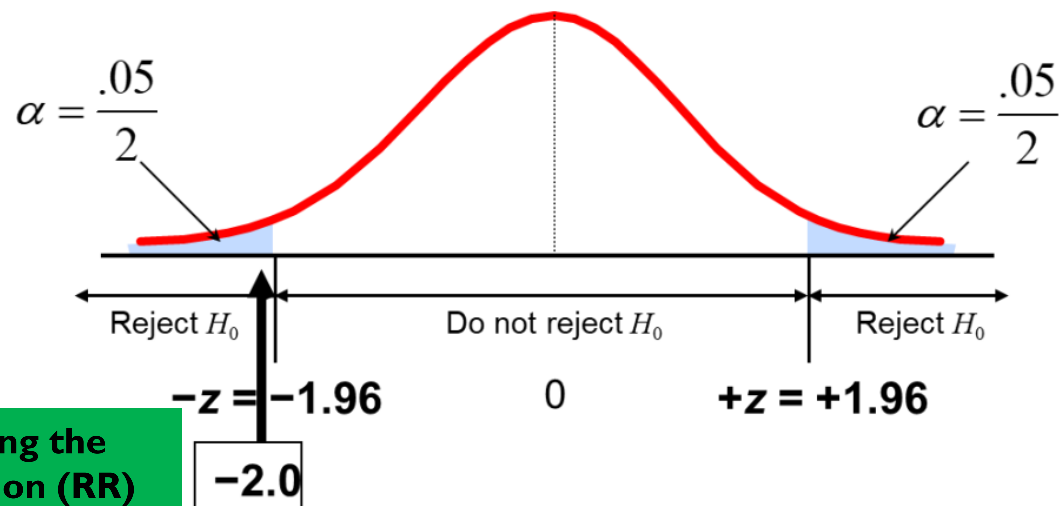
HYPOTHESIS TESTING: EXAMPLE 2

- Reach a decision and interpret the result



Decision Using the Rejection Region (RR)

Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

Newbold et al (2013)

HYPOTHESIS TESTING: EXAMPLE 2

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is $\mu = 3.0$?

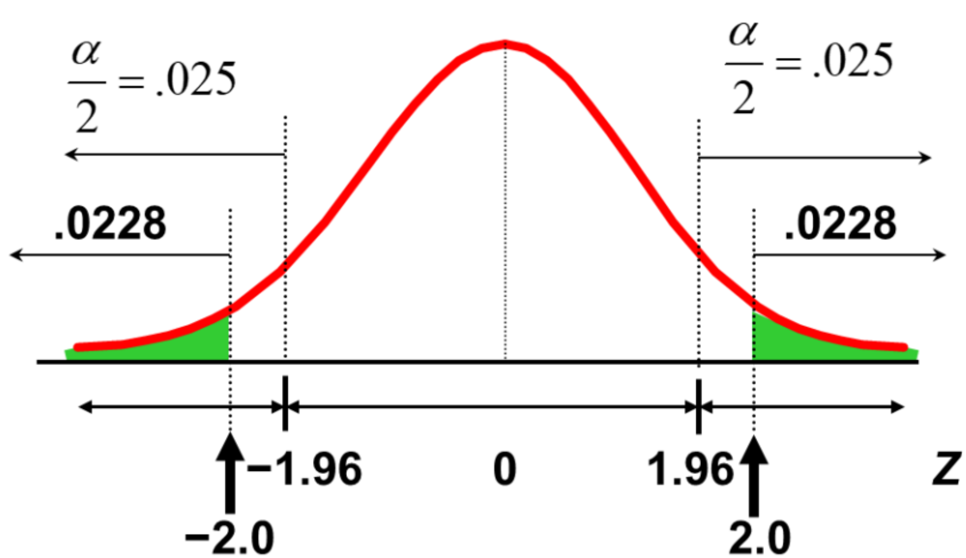
$\bar{x} = 2.84$ is translated to a *z* score of *z* = −2.0

$P(z < -2.0) = .0228$

$P(z > 2.0) = .0228$

***p*-value**

$= \mathbf{.0228 + .0228 = .0456}$



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HYPOTHESIS TESTING: EXAMPLE 2

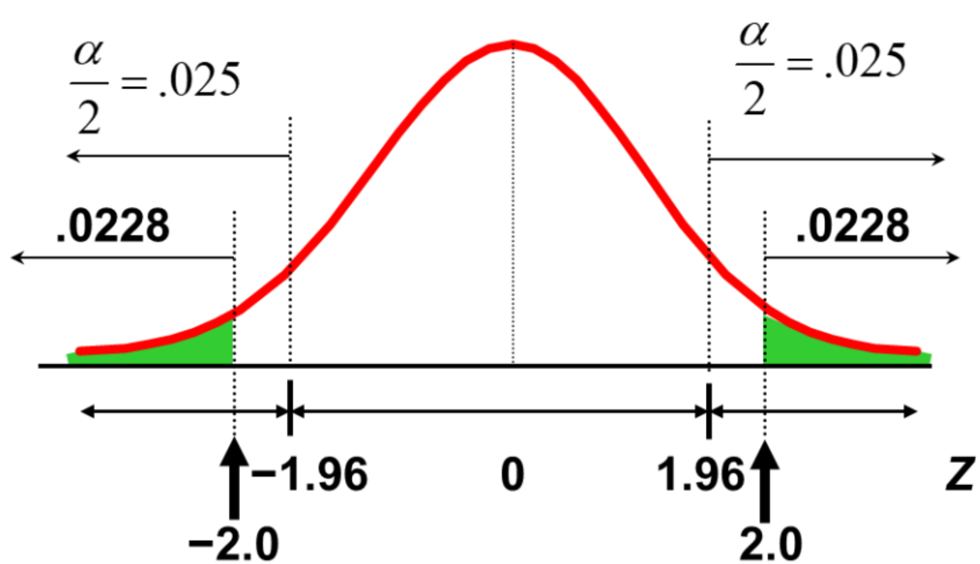
- Compare the *p*-value to $\boldsymbol{\alpha}$
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

Here: p-value $= .0456$

$\alpha = .05$

Decision Using the P-value

Since .0456 < .05, we reject the null hypothesis



Note: Since the p-value is less than $\alpha = 0.05$, we reject H_0 at the 5% significance level.

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HOMEWORK OF LECTURE 13: QUESTIONS

EXERCISE 9.11

9.11 A manufacturer of detergent claims that the contents of boxes sold weigh on average at least 16 ounces. The distribution of weight is known to be normal, with a standard deviation of 0.4 ounce. A random sample of 16 boxes yielded a sample mean weight of 15.84 ounces. Test at the 10% significance level the null hypothesis that the population mean weight is at least 16 ounces.

Newbold et al (2013)

EXERCISE 9.12

9.12 A company that receives shipments of batteries tests a random sample of nine of them before agreeing to take a shipment. The company is concerned that the true mean lifetime for all batteries in the shipment should be at least 50 hours. From past experience it is safe to conclude that the population distribution of lifetimes is normal with a standard deviation of 3 hours. For one particular shipment the mean lifetime for a sample of nine batteries was 48.2 hours. Test at the 10% level the null hypothesis that the population mean lifetime is at least 50 hours.

Newbold et al (2013)

THANKS!

Questions?